CLASSICAL r-MATRICES AND COMPATIBLE POISSON STRUCTURES FOR LAX EQUATIONS ON POISSON ALGEBRAS

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Abstract. Given a classical r-matrix on a Poisson algebra, we show how to construct a natural family of compatible Poisson structures for the Hamiltonian formulation of Lax equations. Examples for which our formalism applies include the Benny hierachy, the dispersionless Toda lattice hierachy, the dispersionless KP and modified KP hierachies, the dispersionless Dym hierachy etc.

1. Introduction.

Two Poisson brackets on the same manifold are said to be compatible if their sum is also a Poisson bracket [GDO, M]. There are many examples of integrable systems which are Hamiltonian with respect to two compatible Poisson structures (see, e.g. [DO]). Indeed, when one of the structures happens to be nondegenerate, there is a simple way which allows one to produce a whole family of compatible Poisson structures [KR, RSTS1]. However, the existence of further structures is not a necessity when the two compatible structures are both degenerate.

In the late seventies, we saw the beginning of the Lie algebraic approach to integrable systems [K, A]. The Korteweg de-Vries (KdV) equation, for example, was shown to be a Hamiltonian system on coadjoint orbits [A]. Furthermore, the second Poisson structure for KdV type equations was constructed on subspaces of the algebra of formal pseudo-differential operators [A, GD]. We now refer to this second

structure as the Adler-Gelfand-Dickey structure. Recently, it was found to be of independent interest in conformal field theory [DFIZ]. In the mean time, the Lie algebraic approach to integrable systems was extensively developed, particularly by the Russian school in St. Petersburg (see, e.g., the survey in [RSTS2]). In the so-called r-matrix framework, the simplest Poisson structures for the Hamiltonian formulation of Lax equations on Lie algebras are the linear Poisson structures associated with the R-brackets. In the case where $\mathfrak{g}$ is the Lie algebra of a non-commutative, associative algebra, a construction of quadratic brackets which give Lax equations was first available for the skew-symmetric r-matrices satisfying the modified Yang-Baxter equation [STS1]. Subsequently, this was superseded by a more general construction valid for a wider class of r-matrices [LP1,LP2]. Indeed, in [LP2], even a third order structure was found. At this juncture, the reader should note that on the abstract level of associative algebras, neither the linear structure nor the quadratic structure is nondegenerate. Therefore, the recipe for producing a whole family of structures is not applicable in this context. As a matter of fact, no Poisson structures with order > 3 was ever found. In this connection, we would like to mention the thesis of Strack [ST], which showed (by using computer algebra) that beyond order 3, no Poisson structures of a certain form can exist for the Hamiltonian formulation of Lax equations. So this is the state of affairs for noncommutative, associative algebras.

In this paper, we address the Hamiltonian formulation of Lax equations, as before, but in the context of Poisson algebras. Here, we show how to construct a natural family of compatible Poisson structures on the full algebra. On the group of invertible elements (if non-empty and forms an open subset), similar consideration shows we can even define structures of negative order. Thus the situation for Poisson algebras, in which multiplication is commutative, is entirely different. Recall that a Poisson algebra is by definition a commutative, associative algebra with

unit 1 equipped with a Lie bracket such that the Leibniz rule holds [W1]. The most familiar examples of Poisson algebras are given by the collection of smooth functions on Poisson manifolds. For us, the particular examples which have partly motivated this work are the algebras associated with the truncated Benney's equation [G-KR], and the various dispersionless equations [DM, K, TT] which are currently of interest in topological field theory [D, K]. As the reader will see, a family of vector fields V_n, $n \geq -1$, plays the key role in this investigation. These vector fields V_n are invariants of degree 1 of the vector fields associated with the Lax equations, and satisfy the Virasoro relations $[V_m, V_n] = (n-m)V_{m+n}$. For a given classical r-matrix on the Poisson algebra, we can construct the associated linear bracket. If we denote by π_{-1} the bivector field corresponding to this basic linear structure, we shall show that the Lie derivatives $L_{V_m}\pi_{-1}$ essentially generate all higher order structures. Thus our construction works for an arbitrary classical r-matrix! This is in marked contrast to previous results on quadratic Poisson structures on non-commutative, associative algebras [STS1, LP1, LP2], where one has to make rather stringent assumptions on the r-matrix. In this connection, we would like to remind the reader of the important difference between the notions of double Lie algebras and Lie bialgebras. Recall that the former was motivated by the study of integrable systems [STS1] and is associated with classical r-matrices. On the other hand, the notion of Lie bialgebras had its origin in the geometry of Poisson Lie groups [DR]. The two do intersect, for example, in the class of double Lie algebras called Baxter Lie algebras [STS2] (where the r-matrix satisfies additional properties). In our case, as the r-matrix is assumed to be completely arbitrary, we are working within the framework of double Lie algebras here.

The paper is organized as follows. In sec. 2, we assemble a number of basic facts and definitions which will be used in the paper. In Sec. 3, we formulate the main result and display the explicit formulas for the linear, quadratic, and higher order

structures. Then we study a number of basic properties. In order to prepare for the proof of the main result, we introduce the vector fields V_n in Sect. 4 and discuss their relation with the Lax equations. Then, in Sect. 5, we give a proof of the main result. In order to illustrate the use of our construction in Sect. 3, we describe the multi-Hamiltonian formalism of some concrete partial differential equations in Sect. 6. Our examples include the hierachy of truncated Benny equations [B, G-KR] in nonlinear waves, the dispersionless Toda lattice hierachy [DM], the dispersionless KP [K, TT] and modified KP hierachies, and the dispersionless Dym hierachy. Note that in each example, the set of Lax operators under consideration is a submanifold of the full Poisson algebra. However, this submanifold is not necessarily a Poisson submanifold of the full algebra equipped with a bracket which comes from Sect. 3. For this reason, the passage from the bracket on the algebra to the Hamiltonian structure on the submanifold of Lax operators might involve the process of reduction [MR]. Thus in our examples, we find Dirac reduction [D, MR] (i.e. reduction with constraints) comes in naturally. For the Benny hierachy and the dispersionless Toda lattice hierachy, we shall compute the first few Poisson structures explicitly, and illustrate the use of Dirac reduction. Our explicit expressions for the structures not only allow us to find the Casimir functions, they also show that the structures which come from our Poisson algebras are of hydrodynamic type or its generalizations [DN, F]. Indeed, as it turns out, all the higher structures of the dispersionless Toda lattice hierachy are nonlocal generalizations of brackets of hydrodynamic type. This shows how our construction in Sect. 3 can get complicated upon reduction to a specific submanifold of Lax operators.

To close we stress again that our main result is formulated along the lines of the r-matrix approach (where Poisson structures are defined either on Lie algebras or their duals, or on Lie groups) and applies to all Poisson algebras satisfying the assumptions of Theorem 3.2. In any concrete applications, the use of reduction

techniques (where necessary) is perfectly natural and the reader should not feel uncomfortable under such circumstances.

2. Preliminaries.

We collect in this section a number of basic facts, and introduce some terminology which will be used in the sequel.

Let P be a smooth manifold. A Poisson bracket $\{\cdot,\cdot\}$ on P is a Lie bracket on $C^\infty(P)$ which satisfies the derivation property in each argument. If π is the bivector field corresponding to the bracket operation, i.e.

$$\{F, H\} = \pi(dF, dH)\,, \tag{2.1}$$

then it is well-known that the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to $[\pi,\pi]_S = 0$ [W2], where $[\cdot,\cdot]_S$ is the Schouten bracket [S]. Recall that if $\Gamma(\wedge^k TM)$ is the space of sections of the vector bundle $\wedge^k TM$, and $\wedge^*(M) = \bigoplus_{k\geq 0} \Gamma(\wedge^k TM)$, the Schouten bracket $[\cdot,\cdot]_S$ is the bilinear map

$$[\cdot,\cdot]_S : \wedge^*(M) \times \wedge^*(M) \to \wedge^*(M) \tag{2.2}$$

which extends the usual Lie bracket operation on $\Gamma(TM)$ and makes $\wedge^*(M)$ into a Lie superalgebra. In particular, the following graded Jacobi identity holds:

$$(-1)^{pr}[u,[v,w]_S]_S + (-1)^{qp}[v,[w,u]_S]_S + (-1)^{rq}[w,[u,v]_S]_S = 0 \tag{2.3}$$

where $u \in \Gamma(\wedge^p TM)$, $v \in \Gamma(\wedge^q TM)$ and $w \in \Gamma(\wedge^r TM)$.

As we mentioned in the introduction, two Poisson brackets on P are said to be compatible if their sum is also a Poisson bracket, i.e. satisfies the Jacobi identity [GDO, M]. In terms of the corresponding bivector fields π_1 and π_2, this is equivalent to $[\pi_1,\pi_2]_S = 0$, as $[\pi_i,\pi_i]_S = 0$, $i = 1, 2$.

In this paper, we shall construct compatible Poisson structures for the Hamiltonian formulation of Lax equations (associated with r-matrices) when the underlying manifold P is a Poisson algebra.

Definition 2.4. Let A be a commutative, associative algebra with unit 1. If there is a Lie bracket on A such that for each element $a \in A$, the operator $ad_a : b \mapsto [a, b]$ is a derivation of the multiplication, then $(A, [\cdot, \cdot])$ is called a Poisson algebra.

Thus the Poisson algebras are Lie algebras with an additional associative algebra structure (with commutative multiplication and unit 1) related by the derivation property to the Lie bracket. Note that some authors call the Lie bracket on A the Poisson structure on A (see, for example, [W1]), but we shall refrain from such usage in order to avoid confusion.

We now recall the notion of a classical r-matrix [STS1]. Let $\mathfrak{g}$ be a Lie algebra. A linear operator R in the space $\mathfrak{g}$ is called a classical r-matrix if the R-bracket given by

$$[X, Y]_R = \frac{1}{2}\left([RX, Y] + [X, RY]\right), \qquad X, Y \in \mathfrak{g} \tag{2.5}$$

is a Lie bracket, i.e. satisifes the Jacobi identity. Some well-known sufficient conditions for $R \in \mathrm{End}(\mathfrak{g})$ to be a classical r-matrix are the Yang-Baxter equation and the modified Yang-Baxter equation. But in this paper, we can establish our results without assuming these conditions.

To close this section, we define what we mean by Lax equations.

Definition 2.6. Let A be a Poisson algebra, and suppose $R \in \mathrm{End}(A)$ is a classical r-matrix. Equations of the form

$$\dot{L} = [R(X(L)), L], \qquad L \in A \tag{2.7}$$

where $X : A \to A$ is a smooth map satisfying

$$[X(L), L] = 0, \qquad dX(L) \cdot [L', L] = [L', X(L)], \qquad L, L' \in A \tag{2.8}$$

are called Lax equations.

The basic Lax equations on A are given by

$$\dot{L} = Z_m(L) = [R(L^m), L], \qquad m \geq 1\,. \tag{2.9}$$

More generally, if H is a smooth ad-invariant function (in the sense defined in (3.1)), then $\dot{L} = [R(L^m dH(L)), L]$ is also a Lax equation.

3. A Family of Compatible Poisson Structures on Poisson Algebras.

In what follows, we shall assume the Poisson algebra A is equipped with a non-degenerate ad-invariant pairing $(\cdot,\cdot)$. A function F defined on A is said to be smooth if there exists a map $dF : A \to A$ such that

$$\frac{d}{dt}\Big|_{t=0} F(L + tL') = (dF(L), L') \quad , \quad L, L' \in A \tag{3.1}$$

Theorem 3.2. *Let A be a Poisson algebra with Lie bracket $[\cdot,\cdot]$ and non-degenerate ad-invariant pairing $(\cdot,\cdot)$ with respect to which the operation of multiplication is symmetric, i.e. $(XY, Z) = (X, YZ)$, $\forall\, X, Y, Z \in A$. Assume $R \in End(A)$ is a classical r-matrix, then*

(a) *for each integer $n \geq -1$, the formula*

$$\{F, H\}_{(n)}(L) = (L, [R(L^{n+1}dF(L)), dH(L)] + [dF(L), R(L^{n+1}dH(L))]) \tag{3.3}$$

(where F and H are smooth) defines a Poisson structure on A,

(b) *the structures $\{\cdot,\cdot\}_{(n)}$ are compatible with each other,*

(c) *if π_n is the bivector field corresponding to $\{\cdot,\cdot\}_{(n)}$ and $D_{\pi_n} : \wedge^*(A) \to \wedge^*(A)$ is the associated coboundary operator, i.e. $D_{\pi_n} X = [\pi_n, X]_S$, $X \in \wedge^*(A)$. There exists vector fields V_m on A, $m \geq -1$ satisfying the Virasoro relations $[V_m, V_n] = (n-m)V_{m+n}$ such that $D_{\pi_n} V_m = (n-m)\pi_{m+n}$, $m, n \geq -1$.*

□

We shall prove this result in Section 5, after we introduce the vector fields V_m in Section 4 and explain what they are in relation to the Lax equations. As the reader

will see, the relations $[\pi_n, V_m]_S = (n-m)\pi_{m+n}$ between the bivector fields which we establish at the beginning of Section 5 play the key role in proving parts (a) and (b) of the above theorem. They are also responsible for the following.

Corollary 3.4. *(Involution of Casimir Functions)* $\{H^0_{\pi_n}(A), H^0_{\pi_n}(A)\}_{(m+n)} = 0$, $m, n \geq -1$. $m \neq n$.

Proof. This follows from the formula $[\pi_n, V_m]_S(dF, dH) = L_{V_m}\pi_n(dF, dH) = V_m\{F, H\}_{(n)} - \{V_m F, H\}_{(n)} - \{F, V_m H\}_{(n)}$. □

Remark 3.5. Note that from the compatibility of the structures, it follows that

$$\{H^0_{\pi_m}(A), H^0_{\pi_m}(A)\}_{(n)} \subset H^0_{\pi_m}(A)\,. \tag{3.6}$$

We now give a number of basic properties of the Poisson structures $\{\cdot,\cdot\}_{(n)}$, $n \geq -1$.

Theorem 3.7. *(a) Smooth functions in A which are ad-invariant Poisson commute in $\{\cdot,\cdot\}_{(n)}$.*

(b) The Hamiltonian system generated by a smooth ad-invariant function H in the Poisson structure $\{\cdot,\cdot\}_{(n)}$ is given by the Lax equation $\dot{L} = [R(L^{n+1}dH(L)), L]$.

Proof. (a) If F and H are smooth functions in A which are ad-invariant, we have $[dF(L), L] = [dH(L), L] = 0$. Therefore, $\{F, H\}_{(n)}(L) = ([dH(L), L], R(L^{n+1}dF(L))) + ([L, dF(L)],\ R(L^{n+1}dH(L))) = 0$.

(b) If H is ad-invariant, for any smooth F, we have $\{F, H\}_{(n)}(L) = (L, [dF(L), R(L^{n+1}dH(L))]) = (dF(L),\ [R(L^{n+1}dH(L)), L])$. □

¿From formula (3.3), it is clear that the bracket $\{\cdot,\cdot\}_{(n)}$ vanishes at the unit 1. Therefore, the linearization of $\{\cdot,\cdot\}_{(n)}$ defines a Lie bracket on A, and an easy calculation shows it coincides with the R-bracket $[\cdot,\cdot]_R$.

The following result is reminiscent of the multiplicative property of Poisson Lie groups [DR]. However, it is in the context of a Poisson algebra and the reason for its validity is entirely different.

Theorem 3.8. *Equip A with the structure $\{\cdot,\cdot\}_{(0)}$ and $A \times A$ with the product structure. Then the multiplication map $m : A \times A \to A$ is a Poisson map.*

Proof. Let F and H be smooth functions on A. For $L_1, L_2 \in A$, let $L = m(L_1, L_2)$. Clearly, $F \circ m$ depends on two variables and by taking its derivative with respect to the i-th variable, $i = 1, 2$, we obtain $d_1(F \circ m)(L_1, L_2) = L_2 dF(L)$, $d_2(F \circ m)(L_1, L_2) = L_1 dF(L)$. To simplify notation, let $X_1 = dF(L)$, $X_2 = dH(L)$ and denote the product structure on $A \times A$ also by $\{\cdot,\cdot\}_{(0)}$, then we have

$$\begin{aligned} (*) \qquad & \{F \circ m, H \circ m\}_{(0)}(L_1, L_2) \\ & = (L_1, [R(LX_1), L_2X_2] + [L_2X_1, R(LX_2)]) \\ & + (L_2, [R(LX_1), L_1X_2] + [L_1X_1, R(LX_2)]) \,. \end{aligned}$$

By the derivation property of $[\cdot,\cdot]$, the commutativity of multiplication and its symmetry with respect to the ad-invariant pairing $(\cdot,\cdot)$, we have

$$\begin{aligned} & (L_1, [R(LX_1), L_2X_2]) \\ & = (L, [R(LX_1), X_2]) - (L_2, [R(LX_1), L_1X_2]) \,. \end{aligned}$$

Likewise,

$$\begin{aligned} & (L_2, [L_1X_1, R(LX_2)]) \\ & = (L, [X_1, R(LX_2)] - (L_1, [L_2X_1, R(LX_2)]) \,. \end{aligned}$$

When we insert these relations in (*), the result follows. □

Consider now A_{inv}, the group of invertible elements of A. We assume $A_{\text{inv}} \neq \phi$ and form an open subset of A. Then we can define vector fields Z_{-m}, V_{-n} for $m \geq 1$, $n \geq 2$, on A_{inv} as in formulas (4.2) and (4.5). If we define

(3.9)
$$\{F, H\}_{(-n)}(L) = (L, [R(L^{-n+1}dF(L)), dH(L)] + [dF(L), R(L^{-n+1}dH(L))]), \quad n \geq 2$$

for smooth functions F and H on A_{inv}, it is easy to check that the analysis in Section 5 also holds for these objects. In particular, this means $\{\cdot,\cdot\}_{(-n)}$ are Poisson structures on A_{inv}.

Theorem 3.10. *Let $\iota : A_{\text{inv}} \to A_{\text{inv}}$ be the inversion map, i.e. $\iota(L) = L^{-1}$. Then $\{F \circ \iota, H \circ \iota\}_{(n)}(L) = -\{F, H\}_{(-n)} \circ \iota(L)$, $n \geq 0$, for all smooth functions F and H on A_{inv}.*

Proof. We have $d(F \circ \iota)(L) = -L^{-2} dF(L^{-1})$ and so
$\{F \circ \iota, H \circ \iota\}_{(n)}(L) = (L, [R(L^{n-1} dF(L^{-1})),\ L^{-2} dH(L^{-1})] - (F \leftrightarrow H))$. Now,

$$
\begin{aligned}
&(L,\ [R(L^{n-1} dF(L^{-1})), L^{-2} dH(L^{-1})]) \\
=&(L,\ L^{-2}[R(L^{n-1} dF(L^{-1})), dH(L^{-1})]) + (L\, dH(L^{-1}),\ [R(L^{n-1} dF(L^{-1})),\ L^{-2}]) \\
=&(L^{-1},\ [R(L^{n-1} dF(L^{-1})),\ dH(L^{-1})]) + 2(dH(L^{-1}),\ [R(L^{n-1} dF(L^{-1})),\ L^{-1}]) \\
=&-(L^{-1}, [R(L^{n-1} dF(L^{-1})),\ dH(L^{-1})])\,.
\end{aligned}
$$

Hence the assertion follows. □

4. Lax Equations on Poisson Algebras and Virasoro Invariants.

According to Definition 2.6, corresponding to each smooth map $X : A \to A$ satisfying (2.8) is a Lax equation

$$
\dot{L} = \widetilde{X}(L) = [R(X(L)), L] \tag{4.1}
$$

To prepare for the proof of Theorem 3.2, we shall introduce vector fields V_n, $n \geq -1$ on A which are related to the Lax equations. Before we do so, we first prove

Theorem 4.2. *Let $X, Y : A \to A$ be smooth maps satisfying (2.8). Then $[\widetilde{X}, \widetilde{Y}] = 0$.*

Proof. We have

$$
\begin{aligned}
&d\widetilde{Y}(L) \cdot \widetilde{X}(L) \\
=&[R(dY(L) \cdot \widetilde{X}(L)), L] + [R(Y(L)),\ \widetilde{X}(L)] \\
=&[R([R(X(L)),\ Y(L)]), L] + [R(Y(L)),\ [R(X(L)), L]]\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&[\widetilde{X},\widetilde{Y}](L)\\
=&2[R([X(L),Y(L)]_R),\ L]+[R(Y(L)),\ [R(X(L)),L]]-[R(X(L)),\ [R(Y(L)),L]]\\
=&[2R([X(L),Y(L)]_R),\ L]-[[R(X(L)),\ R(Y(L))],\ L]\,,\ \text{by Jacobi identity}\\
=&-[[R(X(L)),\ R(Y(L))]-2R([X(L),Y(L)]_R),\ L]\,.
\end{aligned}
$$

Let $B_R(X,Y)=[RX,RY]-2R([X,Y]_R)$. Then R is a classical r-matrix iff $[B_R(X,Y),Z]+[B_R(Y,Z),X]+[B_R(Z,X),Y]=0,\ \forall\ X,Y,Z\in A$. Using the ad-invariant pairing, this is equivalent to

$$
\begin{aligned}
[B_R(X,Y),Z]=&R^*[RX,[Y,Z]]-R^*[X,R^*[Y,Z]]-[RX,R^*[Y,Z]]\\
&+R^*[RY,[Z,X]]-R^*[Y,R^*[Z,X]]-[RY,R^*[Z,X]]
\end{aligned}
$$

If we now put $X=X(L)$, $Y=Y(L)$ and $Z=L$ in the above relation, we obtain $[\widetilde{X},\widetilde{Y}](L)=0$, as asserted. $\square$

The vector fields V_n, $n\geq -1$, are defined as follows:

$$V_n(L)=L^{n+1}\,,\qquad n\geq -1 \tag{4.3}$$

Theorem 4.4. *The vector fields V_n satisfy the Virasoro relations $[V_m,V_n]=(n-m)V_{m+n}$, $m,n\geq -1$.*

Proof. Clear. $\square$

Given a smooth manifold M and a vector field V on M, recall that a tensor field T is an invariant tensor field of V iff $L_V T=0$. Generalizing one step further, we shall say that T is an invariant tensor field of degree 1 iff $L_X^2 T=0$. The vector fields V_n introduced in (4.3) above are invariants of degree 1 of the vector fields $\widetilde{X}$ corresponding to the Lax equations. Indeed, we have

Theorem 4.5. *If $X:A\to A$ is a smooth map satisfying (2.8), we have $L_{V_m}\widetilde{X}=\widetilde{Y}$ where $Y(L)=dX(L)\cdot V_m(L)$.*

Proof.

$$
\begin{aligned}
&[V_m, \widetilde{X}](L)\\
=&d\widetilde{X}(L)\cdot V_m(L) - dV_m(L), \widetilde{X}(L)\\
=&[R(dX(L)\cdot V_m(L)), L] + [R(X(L)), V_m(L)] - (m+1)L^m[R(X(L)), L]\\
=&[R(dX(L), V_m(L)), L]\,.
\end{aligned}
$$

Thus, it remains to show $Y(L) = dX(L)\cdot V_m(L)$ satisfies (2.8). To do this, first note that from the condition $[X(L), L] = 0$, $L \in A$, we have $[dX(L)\cdot L', L] + [X(L), L'] = 0$, $L, L' \in A$. Therefore, $[Y(L), L] = [dX(L)\cdot V_m(L), L] = -[X(L), V_m(L)] = -(m+1)L^m[X(L), L] = 0$, for all $L \in A$. On the other hand, it follows from $dX(L)\cdot [L', L] = [L', X(L)]$, $L, L' \in A$, that

$$
(*) \qquad (d^2X(L)\cdot L')([L'', L]) + dX(L)\cdot [L'', L'] = [L'', dX(L)\cdot L'], L, L', L'' \in A\,.
$$

Consequently, for all $L, L' \in A$, we have

$$
\begin{aligned}
&dY(L)\cdot [L', L] = (d^2X(L)\cdot [L', L])(V_m(L)) + dX(L)\cdot ((m+1)L^m[L', L])\\
&= (d^2X(L)\cdot V_m(L))([L', L]) + dX(L)\cdot [L', V_m(L)]\\
&= [L', Y(L)], \text{ by } (*).
\end{aligned}
$$

□

Remark 4.6. For the vector fields Z_n in (2.9), we have in particular the relations $L_{V_m} Z_n = nZ_{m+n}$, $m \geq -1$, $n \geq 1$.

If we now combine Theorem 4.5 and Theorem 4.2, the nature of the vector fields V_m is now revealed.

Corollary 4.7. $L^2_{\widetilde{X}} V_n = 0$, $n \geq 0$, $L_{\widetilde{X}} V_{-1} = 0$. □

5. Virasoro Action on the Bivector Fields and Compatibility of the Structures.

The goal of this section is to prove Theorem 3.2. To do this, we consider the action of the vector fields V_m on the bivector fields π_n corresponding to $\{\cdot,\cdot\}_{(n)}$, $n \geq -1$.

Theorem 5.1. $L_{V_m}\pi_n = (n-m)\pi_{m+n}$, $m, n \geq -1$.

As indicated in Section 3, this result is the key in proving Theorem 3.2. The demonstration of Theorem 5.1 is quite tedious, so we break it up into several steps. First, note that from the property of the Lie derivative, we have

$$L_{V_m}\pi_n(dF, dH) = V_m\{F, H\}_{(n)} - \{V_m F, H\}_{(n)} - \{F, V_m H\}_{(n)}\,. \tag{5.2}$$

Using the expressions for $\{\cdot,\cdot\}_{(n)}$ and V_m, we obtain the identities in the next two lemmas. We shall omit the rather lengthy computations.

Lemma 5.3.

$$\begin{aligned} &V_m\{F, H\}_{(n)}(L) = (V_m(L), [R(L^{n+1}dF(L)), dH(L)]) \\ &+ (L, [R(L^{n+1}dF(L)), d^2H(L)\cdot V_m(L)]) + (n+1)(L, [R(L^{m+n+1}dF(L)), dH(L)]) \\ &+ (L, [R(L^{n+1}d^2F(L)\cdot V_m(L)), dH(L)]) - (F \leftrightarrow H) \end{aligned}$$

where $(F \leftrightarrow H)$ *denote terms obtained from previous ones by switching* F *and* H.

□

Lemma 5.4.

$$\begin{aligned} &\{V_m F, H\}_{(n)}(L) + \{F, V_m H\}_{(n)}(L) = (L, [R(L^{n+1}d^2F(L)\cdot V_m(L))\,, \\ &dH(L)] + [d^2F(L)\cdot V_m(L), R(L^{n+1}dH(L))]) + (m+1)(L, [R(L^{m+n+1}dF(L)), dH(L)] \\ &+ [L^m dF(L), R(L^{n+1}dH(L))]) - (F \leftrightarrow H)\,. \end{aligned}$$

□

Proof of Theorem 5.1. By combining the expressions in Lemma 5.3 and Lemma 5.4 according to (5.2), it is clear that terms involving second derivatives cancel out, and we obtain

$$\begin{aligned} (*)\qquad &L_{V_m}\pi_n(L)\;(X_1, X_2) \\ &= (V_m(L), [R(L^{n+1}X_1), X_2]) + (n-m)(L, [R(L^{m+n+1}X_1), X_2]) \\ &- (m+1)(L, [L^m X_1, R(L^{n+1}X_2)]) - (1 \leftrightarrow 2) \end{aligned}$$

where $X_1 = dF(L), X_2 = dH(L)$. Now, by repeated application of the derivation property, the commutativity of multiplication and its symmetry with respect to $(\cdot,\cdot)$, we have

$$\begin{aligned}
&(V_m(L), [R(L^{n+1}X_1), X_2]) - (1 \leftrightarrow 2)\\
&= (L, [R(L^{n+1}X_1), L^m X_2]) - (LX_2, [R(L^{n+1}X_1), L^m]) - (1 \leftrightarrow 2)\\
&= (L, [R(L^{n+1}X_1), L^m X_2]) - m(L^m X_2, [R(L^{n+1}X_1), L]) - (1 \leftrightarrow 2)\\
&= (m+1)(L, [R(L^{n+1}X_1), L^m X_2]) - (1 \leftrightarrow 2)\\
&= (m+1)(L, [L^m X_1, R(L^{n+1}X_2)]) - (1 \leftrightarrow 2)\,.
\end{aligned}$$

If we substitute this in (*), the result follows. □

Remark 5.5. In the case of noncommutative, associative algebra, relations similar to the ones in Theorem 5.1 were obtained in [LP2] for the three structures there.

Corollary 5.6. $[\pi_m, \pi_n]_S = \frac{1}{n+2}[V_{n+1}, [\pi_{-1}, \pi_m]_S]_S + \frac{m-n-1}{n+2}[\pi_{-1}, \pi_{m+n+1}]_S$ *for* $m, n \geq -1$.

Proof. From Theorem 5.1 and the graded Jacobi identity for the Schouten bracket, it follows that

$$\begin{aligned}
&[\pi_m, \pi_n]_S\\
&= -\frac{1}{n+2}\,[\pi_m, [V_{n+1}, \pi_{-1}]_S]_S\\
&= -\frac{1}{n+2}\,[V_{n+1}, [\pi_{-1}, \pi_m]_S]_S + \frac{1}{n+2}\,[\pi_{-1}, [V_{n+1}, \pi_m]_S]_S\\
&= -\frac{1}{n+2}\,[V_{n+1}, [\pi_{-1}, \pi_m]_S]_S + \frac{m-n-1}{n+2}\,[\pi_{-1}, \pi_{m+n+1}]_S
\end{aligned}$$

□

Remark 5.7. The formulation of Corollary 5.6 is motivated by similar considerations in [AvM].

Proof of Theorem 3.2. If we set $m = -1$ in the identity in Corollary 5.5, we find $[\pi_{-1}, \pi_n]_S = -\frac{1}{2(n+2)}\,[V_{n+1}, [\pi_{-1}, \pi_{-1}]_S]_S = 0$, $\forall\; n \geq -1$, as π_{-1} is the bivector

field for the Lie-Poisson structure $\{\cdot,\cdot\}_{(-1)}$. From the same identity, it now follows that $[\pi_m,\pi_n]_S = 0$, $\forall\ m,n \geq -1$. Hence the brackets $\{\cdot,\cdot\}_{(n)}$ define compatible Poisson structures on A. Finally, the assertion in part (c) follows from Theorem 5.1. □

6. Some Examples.

In this section, we look at some concrete examples of partial differential equations which can be realized as Lax equations on Poisson algebras. In each case, we describe the multi-Hamiltonian formalism which follows from our universal construction in Sect. 3. The reader should note that in these applications, we are dealing with Lax operators which form submanifolds of the full Poisson subalgebras under consideration. Although these submanifolds of Lax operators are invariant under the dynamics of the associated Lax equations, however, they are not automatically Poisson submanifolds of the brackets which arise from the general construction in Sect. 3. For this reason, there are two kinds of situations in the examples which follows. In the happy case where the submanifold $\mathcal{M}$ of Lax operators does form a Poisson submanifold of $(A,\{\cdot,\cdot\}_{(n)})$, there is of course an induced structure on $\mathcal{M}$ which can be obtained by simple restriction of $\{\cdot,\cdot\}_{(n)}$ to $\mathcal{M}$. On the other hand, when $\mathcal{M}$ is not a Poisson submanifold of $(A,\{\cdot,\cdot\}_{(n)})$, the reader will see that the geometry in each case warrants the application of Dirac reduction, i.e. reduction with constraints [D, KO, MR]. Thus in this latter case, the brackets which arise from the construction in Sect. 3 serve as the starting point of a reduction process from which the constrained brackets on $\mathcal{M}$ are computed.

In the following, we shall rescale the expression for $\{\cdot,\cdot\}_{(n)}$ by the factor $\frac{1}{2}$.

1° The Benny hierachy.

The Benny equations in nonlinear waves [B] (we shall consider the simplest case

here) are given by the quasi-linear system

$$\begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}_t = \begin{pmatrix} u_0 & 1 \\ u_{-1} & u_0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}_x \tag{6.1}$$

We shall deal with the case where u_0, u_{-1} are smooth functions on the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

Following [G-KR], introduce the algebra A of Laurent polynomials in λ, having the form

$$u(x,\lambda) = \sum_i u_i(x)\lambda^i \,, \tag{6.2}$$

where the coefficients u_i are smooth functions on the circle S^1. With the well-known Lie-bracket defined by

$$[u,v]_{-1} = \frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial \lambda}\,, \qquad u, v \in A\,, \tag{6.3}$$

it is clear that $(A,\ [\cdot,\cdot]_{-1})$ is a Poisson algebra. In [G-KR], the Benny equations are rewritten as a Lax equation in this Poisson algebra. Indeed, (6.1) is equivalent to

$$\frac{dL}{dt} = \left[R\left(\frac{1}{4}L^2\right), L \right]_{-1} \tag{6.4}$$

where the Lax operator L is an element of the Benny manifold

$$\mathcal{M}_{\text{Benny}} = \left\{ L \in A \;\middle|\; L(x,\lambda) = \lambda + u_0(x) + u_{-1}(x)\lambda^{-1} \right\} \tag{6.5}$$

and the r-matrix R is the one associated with the direct sum decomposition

$$A = A_{\geqslant 1} \oplus A_{\leqslant 0} \tag{6.6}$$

into subalgebras

$$A_{\geqslant 1} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i \geqslant 1} u_i(x)\lambda^i \right\} \tag{6.7a}$$

$$A_{\leqslant 0} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i \leqslant 0} u_i(x)\lambda^i \right\} . \tag{6.7b}$$

In view of the representation in (6.4), the quasi-linear system (6.1) is only a member of a hierachy of Lax equations on $\mathcal{M}_{\text{Benny}}$, and this is what we call the Benny hierachy. Note that the Poisson algebra introduced above admits the trace functional

$$\mathrm{tr}_{-1} u = \int u_{-1}(x)dx\,, \qquad u \in A \tag{6.8}$$

(here and below we integrate over S^1)
which satisfies the important property

$$\mathrm{tr}_{-1}[u,v] = 0\,, \qquad u, v \in A\,. \tag{6.9}$$

Therefore, we can equip A with a non-degenerate ad-invariant pairing $(\cdot,\cdot)_{-1}$:

$$(u,v)_{-1} = \mathrm{tr}_{-1}(uv) \quad , \qquad u, v \in A\,. \tag{6.10}$$

Thus we have all the ingredients which are required for the application of Theorem 3.2. Consequently, we have a family of Poisson structures $\{\cdot,\cdot\}_{(n)}$, $n \geq -1$, on A.

It is easy to check that $\mathcal{M}_{\text{Benny}}$ is a Poisson submanifold of $(A, \{\cdot,\cdot\}_{(-1)})$. Therefore, the induced structure on $\mathcal{M}_{\text{Benny}}$ provides the first Poisson structure for the equations in the Benny hierachy [G-KR]. Using $u = (u_0, u_{-1})$ as coordinates on $\mathcal{M}_{\text{Benny}}$, the associated Hamiltonian operator is given explicitly by

$$B_{(-1)}(u) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} , \qquad D = \frac{d}{dx} \tag{6.11}$$

which is apparently well-known to people working in other frameworks (see, for example, [DN]). Clearly, this first structure is degenerate, with Casimirs given by $C_1(u) = \int u_0(x)dx$ and $C_2(u) = \int u_{-1}(x)dx$.

Remark 6.12. One of the advantages in formulating the Benny equations as a Lax equation on A is that it automatically suggests a method of solution, namely, via a

factorization problem on a symplectic diffeomorphism group. The analytic details, however, are nontrivial.

We now turn to the higher structures. Here, it is easy to see that $\mathcal{M}_{\text{Benny}}$ is not a Poisson submanifold of any of the brackets $\{\cdot,\cdot\}_{(n)}$, $n \geqslant 0$. However, we shall see that we can apply Dirac reduction to $\{\cdot,\cdot\}_{(n)}$ with appropriate constraints to obtain the higher structures on $\mathcal{M}_{\text{Benny}}$. We shall illustrate the procedure for $n=0$ and $n=1$, thereby obtaining the second and third Poisson structures on $\mathcal{M}_{\text{Benny}}$.

For $n=0$, the Hamiltonian vector field generated by H is of the form

$$
\begin{aligned}
X_H^{(0)}(u) &= u\Pi_{\leqslant -2}([dH(u),u]_{-1}) - [\Pi_{\leqslant 0}(udH(u)),u]_{-1} \\
&= [\Pi_{\geqslant 1}(udH(u)),u]_{-1} - u\Pi_{\geq -1}([dH(u),u]_{-1})\,.
\end{aligned} \tag{6.13}
$$

If $L \in \mathcal{M}_{\text{Benny}}$, it follows from this formula that the highest order term of $X_H^{(0)}(L)$ in λ is λ^0, while the lowest order is in λ^{-2}. Using $u=(u_0,u_{-1},u_{-2})$ as coordinates on the submanifold $\{\lambda + u_0(x) + u_{-1}(x)\lambda^{-1} + u_{-2}(x)\lambda^{-2} \in A\}$, the operator which gives $X_H^{(0)}(L)$ can be computed explicitly:

$$
\begin{pmatrix} D & u_0 D + u_{0x} & u_{-1}D + u_{-1x} \\ u_0 D & 2u_{-1}D + u_{-1x} & 0 \\ u_{-1}D & 0 & -u_{-1}^2 D - u_{-1}u_{-1x} \end{pmatrix} \tag{6.14}
$$

Therefore, we can apply Dirac reduction with constraint $u_{-2} \equiv 0$ to obtain the second structure on $\mathcal{M}_{\text{Benny}}$:

(6.15)

$$
\begin{aligned}
B_0(u) &= \begin{pmatrix} D & u_0 D + u_{0x} \\ u_0 D & 2u_{-1}D + u_{-1x} \end{pmatrix} - \begin{pmatrix} u_{-1}D + u_{-1x} \\ 0 \end{pmatrix} (-u_{-1}^2 D - u_{-1}u_{-1x})^{-1} (u_{-1}D \ 0) \\
&= \begin{pmatrix} 2 & u_0 \\ u_0 & 2u_{-1} \end{pmatrix} D + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u_{0x} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u_{-1x}
\end{aligned}
$$

Note that this second structure is of hydrodynamic type [DN] because the associated Hamiltonian operator is of the form

$$
B_0^{ij}(u) = g^{ij}(u)D + b_k^{ij}(u)\, u_x^k\,. \tag{6.16}
$$

In this case, the metric which defines the structure (6.15) is non-degenerate where

$$\Delta = u_0^2 - 4u_{-1} \neq 0\,. \tag{6.17}$$

For $n = 1$, i.e. for the bracket $\{\cdot,\cdot\}_{(1)}$, we have a similar formula for the Hamiltonian vector field

$$\begin{aligned} X_H^{(1)}(u) &= u^2\Pi_{\leqslant -2}([dH(u),u]_{-1}) - \big[\Pi_{\leqslant 0}(u^2 dH(u)),u\big]_{-1} \\ &= \big[\Pi_{\geqslant 1}(u^2 dH(u)),u\big]_{-1} - u^2\Pi_{\geqslant -1}([dH(u),u]_{-1})\,. \end{aligned} \tag{6.18}$$

This time, the highest order term of $X_H^{(1)}(L)$ ($L \in \mathcal{M}_{\text{Benny}}$) in λ is still λ^0, but the lowest order term is in λ^{-3}. Therefore, in the coordinates $u = (u_0, u_{-1}, u_{-2}, u_{-3})$, the operator which gives $X_H^{(1)}(L)$ is given by

(6.19)
$$\begin{pmatrix} 2u_0D+u_{0x} & (u_0^2+3u_{-1})D+2u_0u_{0x}+2u_{-1x} & 2u_0u_{-1}D+2u_0u_{-1x}+2u_{-1}u_{0x} & u_{-1}^2D+2u_{-1}u_{-1x} \\ (u_0^2+3u_{-1})D+u_{-1x} & 4u_0u_{-1}D+12u_{-1}u_{0x}+2u_0u_{-1x} & u_{-1}^2D+2u_{-1}u_{-1x} & 0 \\ 2u_0u_{-1}D & u_{-1}^2D & -2u_0u_{-1}^2D-2u_0u_{-1}u_{-1x}-u_{-1}^2u_{0x} & -u_{-1}^3D-2u_{-1}^2u_{-1x} \\ u_{-1}^2D & 0 & -u_{-1}^3D-u_{-1}^2u_{-1x} & 0 \end{pmatrix}$$

To obtain the structure on $\mathcal{M}_{\text{Benny}}$, we have to use Dirac reduction with the constraints $u_{-2} \equiv 0$, $u_{-3} \equiv 0$. Accordingly, we have to invert the lower 2×2 block of (6.19):

$$\begin{aligned} &\begin{pmatrix} -2u_0u_{-1}^2D - 2u_0u_{-1}u_{-1x} - u_{-1}^2u_{0x} & -u_{-1}^3D - 2u_{-1}^2u_{-1x} \\ -u_{-1}^3D - u_{-1}^2u_{-1x} & 0 \end{pmatrix}^{-1} \\ &\quad = -\begin{pmatrix} 0 & \frac{1}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}^2} \\ \frac{1}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}} & -\frac{1}{u_{-1}^2}D^{-1}\frac{u_0}{u_{-1}^2} - \frac{u_0}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}^2} \end{pmatrix} \end{aligned} \tag{6.20}$$

Hence the Hamiltonian operator of the third structure is given by

(6.21)
$$\begin{aligned} B_1(u) &= \begin{pmatrix} 2u_0D+u_{0x} & (u_0^2+3u_{-1})D+2u_0u_{0x}+2u_{-1x} \\ (u_0^2+3u_{-1})D+u_{-1x} & 4u_0u_{-1}D+2u_{-1}u_{0x}+2u_0u_{-1x} \end{pmatrix} \\ &\quad + \begin{pmatrix} 2u_0u_{-1}D+2u_{-1}u_{0x}+2u_0u_{-1x} & u_{-1}^2D+2u_{-1}u_{-1x} \\ u_{-1}^2D+2u_{-1}u_{-1x} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}^2} \\ \frac{1}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}} & -\frac{1}{u_{-1}^2}D^{-1}\frac{u_0}{u_{-1}^2} - \frac{u_0}{u_{-1}^2}D^{-1}\frac{1}{u_{-1}^2} \end{pmatrix} \\ &\qquad \begin{pmatrix} 2u_0u_{-1}D & u_{-1}^2D \\ u_{-1}^2D & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4u_0 & u_0^2+4u_{-1} \\ u_0^2+4u_{-1} & 4u_0u_{-1} \end{pmatrix} D + \begin{pmatrix} 2 & 2u_0 \\ 0 & 2u_{-1} \end{pmatrix} u_{0x} + \begin{pmatrix} 0 & 2 \\ 2 & 2u_0 \end{pmatrix} u_{-1x}\,. \end{aligned}$$

Again, this corresponds to a bracket of hydrodynamic type and the non-degeneracy of the metric is characterized by the same condition in (6.17).

Remark 6.22. (a) Alternatively, on the symplectic leaves of the first structure defined by the conditions $\int u_0(x)dx = \text{const}$, $\int u_{-1}(x)dx = \text{const}$, B_{-1} is invertible and therefore we can compute the recursion operator

$$\mathcal{R} = B_0 B_{-1}^{-1} = \begin{pmatrix} u_0 + u_{0x}D^{-1} & 2 \\ 2u_{-1} + u_{-1x}D^{-1} & u_0 \end{pmatrix} \tag{6.23}$$

¿From this, we can check that $B_1 = \mathcal{R}B_0$.

(b) In principle, one can compute all higher structures explicitly by applying Dirac reduction to $\{\cdot,\cdot\}_{(n)}$ or by using the recursion operator $\mathcal{R}$, but the calculations are quite formidable and we do not know if there exists an efficient way to do this.

2° The dispersionless Toda lattice hierachy.

Let A be the algebra introduced in Example 1°, but now we equip it with the following Lie bracket

$$[u,v]_0 = \lambda\left(\frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial \lambda}\right), \qquad u,v \in A\,. \tag{6.24}$$

Then $(A, [\cdot,\cdot]_0)$ is also a Poisson algebra. The dispersionless Toda lattice hierachy is defined by the Lax equations

$$\frac{dL}{dt} = [\Pi_{\mathfrak{k}}(L^n), L]_0 = -\,[\Pi_{\mathfrak{l}}(L^n), L]_0\,, \quad n = 1,2,\ldots \tag{6.25}$$

where the Lax operator L is an element of the manifold

$$\mathcal{M}_{d\text{Toda}} = \{L \in A \mid L(x,\lambda) = u_1(x)\lambda + u_0(x) + u_1(x)\lambda^{-1}\} \tag{6.26}$$

and $\Pi_{\mathfrak{k}}, \Pi_{\mathfrak{l}}$ are the projection operators relative to the direct sum decomposition

$$A = \mathfrak{k} \oplus \mathfrak{l} \tag{6.27}$$

into subalgebras

$$\mathfrak{k} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i>0} u_i(x)(\lambda^i - \lambda^{-i}) \right\} \tag{6.28a}$$

$$\mathfrak{l} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i\le 0} u_i(x)\lambda^i \right\} . \tag{6.28a}$$

When $n = 1$, the corresponding Lax equation

$$\frac{dL}{dt} = [\Pi_{\mathfrak{k}}(L), L]_0 \Longleftrightarrow \begin{cases} u_{0t} = 4u_1 u_{1x} \\ u_{1t} = u_1 u_{0x} \end{cases} \tag{6.29}$$

These are the dispersionless Toda lattice equations and can be obtained from the periodic Toda lattice ODE system

$$\frac{da_k}{dt} = 2(b_k^2 - b_{k-1}^2)\,, \quad \frac{db_k}{dt} = b_k(a_{k+1} - a_k) \tag{6.30}$$

by taking a continuum (or long wave) limit.

The Poisson algebra $(A, [\cdot,\cdot]_0)$ also has all the ingredients needed for the construction in Theorem 3.2. In this case, the invariant trace is of the form

$$\mathrm{tr}_0 u = \int u_0(x)dx\,, \qquad u \in A \tag{6.31}$$

which gives rise to the non-degenerate ad-invariant pairing $(\cdot,\cdot)_0$:

$$(u,v)_0 = \mathrm{tr}_0(uv)\,, \qquad u, v \in A\,. \tag{6.32}$$

As the r-matrix for the equations in (6.25) is given by

$$R = \Pi_{\mathfrak{k}} - \Pi_{\mathfrak{l}}\,, \tag{6.33}$$

it follows from (3.3) that the Hamiltonian vector field generated by H in the structure $\{\cdot,\cdot\}_{(n)}$ is of the form

$$\begin{aligned} X_H^{(n)}(u) &= \left[\Pi_{\mathfrak{k}}(u^{n+1}dH(u)), u\right]_0 - u^{n+1}\Pi_{\mathfrak{l}}^*([dH(u),u]_0) \\ &= u^{n+1}\Pi_{\mathfrak{k}}^*([dH(u),u]_0) - \left[\Pi_{\mathfrak{l}}(u^{n+1}dH(u)), u\right]_0 . \end{aligned} \tag{6.34}$$

Using this formula, we can now check that $\mathcal{M}_{d\mathrm{Toda}}$ is a Poisson submanifold of $(A, \{\cdot,\cdot\}_{(n)})$ only for $n = -1, 0$, and 1. Accordingly, the induced structures on $\mathcal{M}_{d\mathrm{Toda}}$ provide the first, second and third Poisson structures for the equations in the dispersionless Toda lattice hierachy. Using $u = (u_0, u_1)$ as coordinates on $\mathcal{M}_{d\mathrm{Toda}}$, the Hamiltonian operator of the first structure is given explicitly by

$$B_{-1}(u) = \begin{pmatrix} 0 & u_1 \\ u_1 & 0 \end{pmatrix} D + \begin{pmatrix} 0 & u_{1x} \\ 0 & 0 \end{pmatrix}. \tag{6.35}$$

Clearly, the associated Hamiltonian vector fields preserve the sign of u_1. Therefore, $B_{-1}(u)$ restricts to a structure on

$$\mathcal{M}^+_{d\mathrm{Toda}} = \{L \in A \mid L(x,\lambda) = u_1(x)\lambda + u_0(x) + u_1(x)\lambda^{-1}, \quad u_1(x) > 0\} \tag{6.36}$$

whose symplectic leaves are the level sets of the Casimirs $\int u_0(x)dx$, $\int \ln u_1(x)dx$. Finally, we note that $B_{-1}(u)$ is obviously of hydrodynamic type and the corrre-sponding metric is non-degenerate on $\mathcal{M}^+_{d\mathrm{Toda}}$ as $\det\,(g^{ij}) = -u_1^2$.

Remark 3.37. Note that the equations in the dispersionless Toda lattice hierachy are (strictly) hyperbolic in $\mathcal{M}^+_{d\mathrm{Toda}}$ and we can take

$$w_1(u) = u_0 - 2u_1, \qquad w_2(u) = u_0 + 2u_1 \tag{6.38}$$

as the Riemann invariants. We shall not give the proof as the reader can easily supply the details.

As for the second and third Poisson structures on $\mathcal{M}_{d\mathrm{Toda}}$, direct calculation shows the corresponding Hamiltonian operators have the form

$$B_0(u) = \begin{pmatrix} 4u_1^2 & u_0u_1 \\ u_0u_1 & u_1^2 \end{pmatrix} D + \begin{pmatrix} 0 & 0 \\ u_1 & 0 \end{pmatrix} u_{0x} + \begin{pmatrix} 4u_1 & u_0 \\ 0 & u_1 \end{pmatrix} u_{1x} \tag{6.39}$$

(6.40)

$$B_1(u) = \begin{pmatrix} 8u_0u_1^2 & 4u_1^3 + u_0^2u_1 \\ 4u_1^3 + u_0^2u_1 & 2u_0u_1^2 \end{pmatrix} D + \begin{pmatrix} 4u_1^2 & 0 \\ 2u_0u_1 & u_1^2 \end{pmatrix} u_{0x} + \begin{pmatrix} 8u_0u_1 & u_0^2 + 8u_1^2 \\ 4u_1^2 & 2u_0u_1 \end{pmatrix} u_{1x}.$$

These structures also restrict to $\mathcal{M}^+_{d\mathrm{Toda}}$, and are obviously of hydrodynamic type. But in contrast to the first structure, the metrics associated with $B_0(u)$ and $B_1(u)$ are non-degenerate only on a subset of $\mathcal{M}^+_{d\mathrm{Toda}}$, characterized by the condition

$$w_1(u)w_2(u) \neq 0\,, \tag{6.41}$$

where $w_1(u), w_2(u)$ are the Riemann invariants in (6.38).

In order to compute the higher structures, we have to invoke Dirac reduction, as in the last example. Here, we shall do this for the fourth structure as it presents new features which are also shared by all higher structures. First of all, we check that for $L \in \mathcal{M}^+_{d\mathrm{Toda}}$, we have $X^{(2)}_H(L) \in Im\Pi^*_{\mathfrak{l}}$, and the highest order term in λ is λ^2. Then we write down the operator which gives $X^{(2)}_H(L)$ using the coordinates $u = (u_0, u_1, u_2)$ on the submanifold where $X^{(2)}_H(L)$ lies:

(6.42)
$$\begin{pmatrix} (12u_1^4+12u_0^2u_1^2)D+6(u_1^4+u_0^2u_1^2)_x & (u_0^3u_1+12u_0u_1^3)D+6u_1^3u_{0x}+24u_0u_1^2u_{1x}+u_0^3u_{1x} & 2u_1^4D+8u_1^3u_{1x} \\ (u_0^3u_1+12u_0u_1^3)D+u_1(u_0^3+6u_0u_1^2)_x & (4u_1^4+3u_0^2u_1^2)D+3u_0u_1^2u_{0x}+(8u_1^3+3u_0^2u_1)u_{1x} & u_1^3u_{0x} \\ 2u_1^4D & -u_1^3u_{0x} & -u_1^4D-2u_1^3u_{1x} \end{pmatrix}$$

Finally, we invoke Dirac reduction with constraint $u_2 \equiv 0$ to compute the Hamiltonian operator of the fourth structure, and the result is

(6.43)
$$\begin{aligned} B_2(u) = &\begin{pmatrix} 16u_1^4 + 12u_0^2u_1^2 & 12u_0u_1^3 + u_0^3u_1 \\ 12u_0u_1^3 + u_0^3u_1 & 4u_1^4 + 3u_0^2u_1^2 \end{pmatrix} D + \begin{pmatrix} 12u_0u_1^2 & 4u_1^3 \\ 3u_0^2u_1 + 8u_1^3 & 3u_0u_1^2 \end{pmatrix} u_{0x} \\ &+ \begin{pmatrix} 32u_1^3 + 12u_0^2u_1 & 24u_0u_1^2 + u_0^3 \\ 12u_0u_1^2 & 8u_1^3 + 3u_0^2u_1 \end{pmatrix} u_{1x} \\ &- \begin{pmatrix} 16u_1u_{1x}D^{-1}u_1u_{1x} & 4u_1u_{1x}D^{-1}u_1u_{0x} \\ 4u_1u_{0x}D^{-1}u_1u_{1x} & u_1u_{0x}D^{-1}u_1u_{0x} \end{pmatrix}\,. \end{aligned}$$

Thus, $B_2(u)$ has a nonlocal tail, and provides an example of a class of nonlocal Hamiltonian operators of the form

$$B^{ij}(u) = g^{ij}D + b^{ij}_k u^k_x + \sum_{\alpha=1}^{N} (w^\alpha)^i_k u^k_x D^{-1} (w^\alpha)^j_\ell u^\ell_x\,. \tag{6.44}$$

In the case where $\det(g^{ij}) \neq 0$, we note that the geometric root of such structures was discussed in [F] and applied to the chromatography equations. At this point,

the reader can check that the subset of $\mathcal{M}^+_{d\mathrm{Toda}}$ where the metric associated with $B_2(u)$ is on-degenerate is likewise defined by (6.41). Also, on the symplectic leaves of the first structure where $\int u_0(x)dx = \mathrm{const}$ and $\int \ln u_1(x)dx = \mathrm{const}$, the recursion operator $\mathbb{R} = B_0 B_{-1}^{-1}$ exists and it is not hard to show that $B_1 = \mathcal{R} b_0$ and $B_2 = \mathcal{R}^2 B_0$.

Remark 6.45. In [DM], the authors considered the dispersionless Toda lattice equations with boundary conditions $u_1(0) = 0$, $u_1(1) = 0$. We remark that the multi-Hamiltonian formalism of this problem can also be obtained in a similar fashion. Indeed, the only major change one has to make here is to replace the algebra above by the algebra of Laurent polynomials in λ, having the form $u(x,\lambda) = \sum_i u_i(x)\lambda^i$, where the coefficients u_i are smooth functions on $I = [0,1]$ satisfying the additional conditions $u_j(0) = u_j(1) = 0$, $j \neq 0$. Otherwise, everything goes through just the same as before. In particular, the formula for the Hamiltonian operators of the first four structures are still those given in (6.35), (6.39), (6.40) and (6.43).

In the next two examples, we shall consider equations with infinitely many field variables. For simplicity of exposition, we shall not get into reduction calculations here, only remark that the number of constraints is still finite in each case.

3° The dispersionless KP hierachy.

Let A be the algebra of formal Laurent series in λ, having the form

$$u(x,\lambda) = \sum_{i=-\infty}^{N(u)} u_i(x)\lambda^i\,, \tag{6.46}$$

where the coefficients u_i are smooth functions on $S^1 = \mathbb{R}/\mathbb{Z}$. Define

$$[u,v]_{-1} = \frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial \lambda}\,, \qquad u, v \in A\,, \tag{6.47}$$

then $(A[\cdot,\cdot]_{-1})$ is a Poisson algebra. The (extended) dispersionless KP (dKP) hierachy is defined by the equations

$$\frac{dL}{dt} = [\Pi_{\geq 0}(L^n), L]_{-1} = -\,[\Pi_{\leq -1}(L^n), L]_{-1}\,, \qquad n = 1, 2, \ldots, \tag{6.48}$$

where the Lax operator is an element of the (extended) dKP manifold

$$\mathcal{M}_{dKP} = \left\{ L \in A \;\middle|\; L(x,\lambda) = \lambda + \sum_{i=-\infty}^{0} u_i(x)\lambda^i \right\}, \tag{6.49}$$

and $\Pi_{\geq 0}, \Pi_{\leq -1}$ are projection operators relative to the decomposition

$$A = A_{\geq 0} \oplus A_{\leq -1} \tag{6.50}$$

into subalgebras

$$A_{\geq 0} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i \geq 0} u_i(x)\lambda^i \right\}. \tag{6.51a}$$

$$A_{\leq -1} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i=-\infty}^{-1} u_i(x)\lambda^i \right\}. \tag{6.51b}$$

In the standard form of the dKP equations [TT], the coefficient $u_0 \equiv 0$, but we shall not get into reduction calculations here.

For the Poisson algebra $(A, [\cdot,\cdot]_{-1})$, the invariant trace is defined by

$$\mathrm{tr}_{-1} u = \int u_{-1}(x)dx, \qquad u \in A, \tag{6.52}$$

and we have the non-degenerate ad-invariant pairing $(\cdot,\cdot)_{-1}$:

$$(u,v)_{-1} = \mathrm{tr}_{-1}(uv), \qquad u, v \in A. \tag{6.53}$$

So again we can invoke Theorem 3.2,using the r-matrix

$$R = \Pi_{\geq 0} - \Pi_{\leq -1} \tag{6.54}$$

in this case to obtain the corresponding brackets $\{\cdot,\cdot\}_{(n)}$, $n \geq -1$. Here, it is easy to check that $\mathcal{M}_{dkP}$ is a Poisson submanifold of $(A, \{\cdot,\cdot\}_{(n)})$ only for $n = -1, 0$. Therefore, the induced structures on $\mathcal{M}_{dKP}$ provide the first and second Hamiltonian structures for the equations in the hierachy. For the bracket $\{\cdot,\cdot\}_{(1)}$, the slightly larger manifold $\left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i=-\infty}^{1} u_i(x)\lambda^i \right\}$ is a Poisson submanifold. Hence the third structure on $\mathcal{M}_{dKP}$ can be computed using Dirac reduction with constraint $u_1 \equiv 1$. We shall leave the details to the interested reader.

4° The dispersionless modified KP and the dispersionless Dym hierachy.

Let $(A, [\cdot,\cdot]_{-1})$ be the Poisson algebra in Example 3°, with the same invariant pairing $(\cdot,\cdot)_{-1}$. Consider the decomposition

$$A = A_{\geq k} \oplus A_{\leq k-1}\ , \qquad k \geq 0 \tag{6.55}$$

with associated projection operators $\Pi_{\geq k}$ and $\Pi_{\leq k-1}$, where

$$A_{\geq k} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i \geq k} u_i(x)\lambda^i \right\}\ , \tag{6.56a}$$

$$A_{\leq k-1} = \left\{ u \in A \;\middle|\; u(x,\lambda) = \sum_{i=-\infty}^{k-1} u_i(x)\lambda^i \right\}\ . \tag{6.56b}$$

Clearly, $A_{\geq k}$ is a subalgebra of $(A, [\cdot,\cdot]_{-1})$ for all k. On the other hand, simple verification shows that $A_{\leq k-1}$ is a subalgebra of $(A, [\cdot,\cdot]_{-1})$ only for $k = 0, 1, 2$. Therefore, among the direct sum decompositions in (6.55), only the three cases $k = 0, 1$, and 2 lead to r-matrices, and the case $k = 0$ has already appeared in Example 3°. We now consider the other two cases, with Lax equations

(6.57)
$$\frac{dL}{dt} = [\Pi_{\geq k}(L^n), L]_{-1} = -\,[\Pi_{\leq k-1}(L^n), L]_{-1}\ , \qquad n = 1, 2, \ldots, \ ;\ k = 1, 2, \ .$$

For $k = 1$ and $L \in \mathcal{M}_{dKP}$, the equations in (6.57) constitute the dispersionless modified KP hierachy. For $k = 2$, we obtain the dispersionless Dym hierachy when the Lax operator L is from the submanifold

$$\mathcal{M}_{dDym} = \left\{ L \in A \;\middle|\; L(x,\lambda) = \sum_{i=-\infty}^{1} u_i(x)\lambda^i \right\}\ . \tag{6.58}$$

These hierachies are the semi-classical limit of the modified KP and the Dym hierachies in [ANPV,KO]. For the dmKP hierachy, with r-matrix given by $R = \Pi_{\geq 1} - \Pi_{\leq 0}$, the manifold of Lax operators is a Poisson submanifold of the associated brackets $\{\cdot,\cdot\}_{(n)}$ for $k = -1, 0, 1$. Hence the induced structures on $\mathcal{M}_{dKP}$

provide the first three Poisson structures for the Hamiltonian description of dmKP. The higher structures, on the other hand, have to be computed using Dirac reduction. For the dispersionless Dym hierachy, the situation is even better, for in this case the first five Poisson structures on $\mathcal{M}_{dDym}$ are obtained from the brackets $\{\cdot,\cdot\}_{(n)}$ $(-1 \leq n \leq 3)$ associated with $R = \Pi_{\geq 2} - \Pi_{\leq 1}$ by simple restriction. Again, the passage from $\{\cdot,\cdot\}_{(n)}$ $(n \geq 4)$ to the higher structures require the application of Dirac reduction.

References

[A] Adler, M., *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg de-Vries (KdV) type equations*, Invent. Math. **50** (1979), 219–248.

[AvM] Adler, M., van Moerbeke, P., *Compatible Poisson structures and the Virasoro algebra*, Comm. Pure Appl. Math. **47** (1994), 5–37.

[ANPV] Aratyn, H., Nissimov, E., Pacheva, S., Vaysburd, I., *R-matrix formulation of the KP hierachies and their gauge equivalence*, Phys. Lett. B **294** (1992), 167–176.

[B] Benny, D. J., *Some properties of long nonlinear waves*, Stud. Appl. Math. **52** (1973), 45–50.

[D] Dubrovin, B., *Geometry of 2D topological field theories*, In: Lecture Notes in Math., vol. 1620, Springer-Verlag, 1996.

[DFIZ] DiFrancesco, P., Itzykson, C., Zuber, J.-B., *Classical W-algebras*, Commun. Math. Phys. **140** (1991), 543–567.

[DM] Deift, P., McLaughlin, K. T-R., *A continuum limit of the Toda lattice*, Memoirs of Amer. Math. Soc. **vol.** (1998).

[DN] Dubrovin, B., Novikov, S. P., *Hydrodynamics of weakly deformed soliton lattices, differential geometry and Hamiltonian theory*, Russian Math. Surveys **44** (1989), 35–124.

[DO] Dorfman, I., *Dirac structures and integrability of nonlinear evolution equations*, Chichester, England : J. Wiley, 1993.

[DR] Drinfeld, V. G., *Hamiltonian structure on Lie groups, Lie bialgebras and the geometrical meaning of the Yang-Baxter equations*, Sov. Math. Doklady **27** (1983), 69–71.

[F] Ferapontov, E. V., *Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type*, Funct. Anal. Appl. **25** (1991), 195–204.

[GD] Gelfand, I. Dickey, L., *A family of Hamiltonian structures related to nonlinear integrable differential equations*, Preprint no. 136, Inst. Appl. Math. USSR Acad. Sci. 1978 (in Russian), English transl. in Collected papers of I. M. Gelfand, Vol. 1. Berlin, Heidelberg, New York : Springer 1987, pp. 625–646.

[GDO] Gelfand, I., Dorfman, I., *Hamiltonian operators and algebraic structures related to them*, Funct. Anal. Appl. **13** (1979), 248–262.

[G-KR] Golenischeva-Kutuzova, M., Reiman, A. G., *Integrable equations, related with the Poisson algebra*, J. Soviet Math. **169** (1988), 890–894.

[K] Kostant, B., *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), 195–338.

[Kri] Krichever, I. M., *The dispersionless Lax equations and topological minimal models*, Commun. Math. Phys. **143** (1991), 415–426.

[KO] Konopelchenko, B., Oevel, W., *An r-matrix approach to nonstandard classes of integrable equations*, Publ. RIMS, Kyoto Univ. **29** (1993), 581–666.

[KR] Kulish, P. P., Reiman, A. G., *Hierarchy of symplectic forms for the Schrodinger and the Dirac equations on the line*, Zap. Nauchn. Sem. L. O. M. I. **77** (1978), 134–147, (in Russian), English transl. in J. Soviet Math. 22, 1627–1637 (1983).

[LP1] Li, L. C., Parmentier, S., *A new class of quadratic Poisson structures and the Yang-Baxter equation*, C. R. Acad. Sci., Paris Ser. I 307, 279–281 (1988).

[LP2] Li, L. C., Parmentier, S., *Nonlinear Poisson structures and r-matrices*, Commun. Math. Phys. **125** (1989), 545–563.

[M] Magri, F., *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19** (1978), 1156–1162.

[MR] Marsden, J., Ratiu, T., *Reduction of Poisson manifolds*, Lett. in Math. Phys. **11** (1986), 161–169.

[RSTS1] Reiman, A. G., Semenov-Tian-Shansky, M. A., *A family of Hamiltonian structures, hierarchy of Hamiltonians, and reduction for first-order matrix differential operators*, Funct. Anal. Appl. **14** (1980), 146–148.

[RSTS2] Reiman, A. G., Semenov-Tian-Shansky, M. A., *Group-theoretical methods in the theory of finite dimensional integrable systems*, In: Dynamical Systems VII, ed. by V. I. Arnold, S. P. Novikov, Encyclopaedia of Mathematical Sciences, vol. 16, Springer-Verlag, 1994.

[S] Schouten, J. A., *On the differential operators of first order in tensor calculus*, Conv. di Geom. Differen., 1953, Ed. Cremonese, Roma 1954.

[ST] Strack, K., *r-Matrizen and assoziativen Algebren: eine systematische Suche nach Poisson-Klammern*, thesis (1990).

[STS1] Semenov-Tian-Shansky, M. A., *What is a classical r-matrix?*, Funct. Anal. Appl. 17, 259–272 (1983).

[STS2] Semenov-Tian-Shansky, M. A., *Dressing transformations and Poisson Lie group actions*, Publ. RIMS, Kyoto University **21** (1985), 1237–1260.

[TT] Takasaki, K., Takebe, T., *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. 7, 743–808 (1995).

[W1] Weinstein, A., *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan 40, 705–727 (1988).

[W2] Weinstein, A., *The local structure of Poisson manifolds*, J. Diff. Geom. 18, 523–557 (1983).